MATH2050B 1920 HW Sec 5.3, 5.4
TA's solutions[1] to selected problems

Section 5.3

Q4. Show that every polynomial of odd degree with real coefficients has at least one real root.

Solution. Let p be an odd degree polynomial. Write

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n$$

where $a_n \neq 0$. We may assume $a_n > 0$. Then $\lim_{n\to\infty} p(x) = +\infty$, $\lim_{x\to-\infty} p(x) = -\infty$. Since p is continuous on $\mathbb{R}$, so p has a real root by Intermediate Value Theorem.

Q5. Show that the polynomial $p(x) = x^4 + 7x^3 - 9$ has at least two real roots. Use a calculator to locate these roots to within two decimal places.

Solution. Since $\lim_{x\to\infty} p(x) = +\infty$, $\lim_{x\to-\infty} p(x) = +\infty$, $p(0) < 0$, so p has two real roots.

Q6. Let f be continuous on the interval $[0, 1]$ to $\mathbb{R}$ and such that $f(0) = f(1)$. Prove that there exists a point c in $[0, \frac{1}{2}]$ such that $f(c) = f(c + \frac{1}{2})$. Conclude that there are, at any time, antipodal points on the earth's equator that have the same temperature.

Solution. Let $g(x) = f(x) - f(x + \frac{1}{2})$. Then $g(\frac{1}{2}) = -g(0)$. If $g(0) = 0$, then we can take $c = 0$. Suppose $g(0) \neq 0$, then $g(\frac{1}{2})$, $g(0)$ are of opposite signs, therefore by Intermediate Value Theorem there exists $c \in [0, \frac{1}{2}]$ s.t. $g(c) = 0$, i.e. $f(c) = f(c + \frac{1}{2})$.

We may view the earth's equator as $I = [0, 1]$ identifying 0 and 1. Let f be the function assigning to each point $x \in [0, 1]$ its corresponding temperature. Then $f(0) = f(1)$ and conclusion follows from the above.

Q11. Let $I = [a, b]$, let $f : I \to \mathbb{R}$ be continuous on I, and assume that $f(a) < 0$, $f(b) > 0$. Let $W := \{x \in I : f(x) < 0\}$, and let $w := \sup W$. Prove that $f(w) = 0$.

Solution. Since $w = \sup W$, so there is a sequence $(w_n)_{n=1}^{\infty}$ in W s.t. $w_n \to w$. By continuity $f(w) = \lim_{n\to\infty} f(w_n) \leq 0$. So $f(w) \leq 0$. We claim that $f(w) < 0$ is impossible.

If $f(w) < 0$, then by continuity, there is a small δ' s.t. $f(x) < 0$ for all $w - \delta' < x < w$. This contradicts to $w = \sup W$. (By definition of sup, $\because \delta' > 0$, $\therefore \exists\, y \in W$ s.t. $w - \delta' < y < w$)

Hence $f(w) = 0$.

Q12. Let $I := [0, \pi/2]$ and let $f : I \to \mathbb{R}$ be defined by $f(x) = \sup\{x^2, \cos x\}$ for $x \in I$. Show there exists an absolute minimum point $x_0 \in I$ for f on I. Show that x_0 is a solution to the equation $\cos x = x^2$.

Solution. Let $g(x) = x^2$, $h(x) = \cos x$. The facts we will need is that g is increasing, h is decreasing.

- $g(0) = 0$, $g(\frac{\pi}{2}) = \frac{\pi^2}{4}$.
- $h(0) = 1$, $h(\frac{\pi}{2}) = 0$.

[1]please kindly send an email to `ncliu@math.cuhk.edu.hk` if you have spotted any typo/error/mistake.

So by Intermediate Value Theorem there is x_0 s.t. $g(x_0) = h(x_0)$ (by considering $g - h$). Now it follows that

$$f(x) = \begin{cases} \cos x & x \in [0, x_0] \\ x^2 & x \in (x_0, \frac{\pi}{2}] \end{cases}$$

By monotonicity, $\cos x \geq \cos x_0$ for all $x \in [0, x_0]$. And $x^2 \geq x_0^2$ for all $x \in (x_0, \frac{\pi}{2}]$. Hence $f(x) \geq f(x_0)$ for all $x \in I$.

Q13. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and that $\lim_{x \to -\infty} f = 0$ and $\lim_{x \to \infty} f = 0$. Prove that f is bounded on $\mathbb{R}$ and attains either a maximum or minimum on $\mathbb{R}$. Give an example to show that both maximum and a minimum need not be attained.

Solution. Consider $\epsilon_0 = 1$. By continuity there is a large M s.t. $|f(x)| < 1$ on $\mathbb{R} \setminus [-M, M]$. f must be bounded on the closed and bounded interval $[-M, M]$. Thus f is bounded on $\mathbb{R}$.

To show that f has a max or min, we split into cases.

Case 1. $f = 0$ on $\mathbb{R}$. In this case f certainly attains a max.

Case 2. $f(x_0) > 0$ for some $x \in \mathbb{R}$. By continuity there is a large M' s.t. $|f(x)| < \frac{f(x_0)}{2}$ for all $x \in \mathbb{R} \setminus [-M', M']$. Note f attains a maximum on $[-M', M']$, say $f(x_1) = \max_{x \in [-M', M']} f(x)$.

Since $x_0 \in [-M', M']$, so $f(x_1) \geq f(x_0)$. It follows that $f(x_1) \geq f(x)$ for all $x \in \mathbb{R}$.

Case 3. $f(x_0) < 0$ for some $x \in \mathbb{R}$. (Similar to Case 2)

Example (of a function that only one of max or min is attained) $f(x) = e^{-x^2}$. f only attains a maximum at 0, with no minimum. ($f > 0$ on $\mathbb{R}$)

Section 5.4

Q10. Prove that if f is uniformly continuous on a bounded subset A of $\mathbb{R}$, then f is bounded on A.

Solution. Suppose f is not bounded on A, then there is a sequence (x_n) in A s.t. $|f(x_n)| > n$ for all n. By BW Theorem (x_n) has a convergent subsequence (x_{n_k}). By uniform continuity, $(f(x_{n_k}))_{k=1}^{\infty}$ must be Cauchy. But $(f(x_{n_k}))_{k=1}^{\infty}$ is unbounded, contradiction.

Q14. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be **periodic** on $\mathbb{R}$ if there exists a number $p > 0$ s.t. $f(x + p) = f(x)$ for all $x \in \mathbb{R}$. Prove that a continuous periodic function on $\mathbb{R}$ is bounded and uniformly continuous on $\mathbb{R}$.

Solution. It is easy to deduce $f(x) = f(x + kp)$ for all $x \in \mathbb{R}$ and for all $k \in \mathbb{Z}$. Note that

$$\sup_{x \in \mathbb{R}} |f(x)| = \sup_{x \in [0,p]} |f(x)|$$

Because $[0, p]$ is a closed and bounded interval, so f is bounded on $[0, p]$ and hence on $\mathbb{R}$. We need to show f is uniformly continuous on $\mathbb{R}$.

Let $\epsilon > 0$.

Because f is uniformly continuous on $[-p, 2p]$, there is $\delta > 0$ s.t. for all $x, y \in [-p, 2p]$, $|x - y| < \delta$, $|f(x) - f(y)| < \epsilon$. Now, for all $x, y \in \mathbb{R}$, $|x - y| < \delta$, we can find an integer $k \in \mathbb{Z}$ s.t. $x + kp \in [0, p]$. If δ is small enough then $y + kp \in [-p, 2p]$. Therefore

$$|f(x) - f(y)| = |f(x + kp) - f(y + kp)| < \epsilon.$$